Application of Convex Relaxation to Array Synthesis Problems

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***Abstract*— A general procedure to solve efficiently non convex array synthesis problems is presented. It is based on the SemiDefinite Relaxation (SDR) technique. The way to properly relax the constraints in order to formulate the synthesis of shaped beams, phase-only arrays and reconfigurable arrays as semidefinite programming problems is detailed. These so-approximated array synthesis problems are then convex, easy to implement and can be efficiently solved using off-the-shelf numerical routines. The conditions under which the relaxed problems provide the optimal solution to the original non convex synthesis problems are specified. Various representative numerical comparisons with arrays designed by other approaches show the validity of the proposed method and illustrate its potentialities.**

***Index Terms*— Antenna synthesis, array antennas, shaped beam, phase-only control, reconfigurable arrays, convex optimization.**

I. Introduction

A HOST of applications ranging from radar and remote sensing to communication systems require the design of efficient antenna arrays. This is the reason why the research field of array synthesis has received a lot of attention since the fifties.

Among the array synthesis problems, a large number are difficult optimization problems because of their non convexity. Let us cite, for instance, the synthesis of:

- shaped beams where the desired power radiated by the array is both upper and lower bounded,
- phase-only arrays in which the excitation magnitudes are known and fixed and only the phases are optimized,
- reconfigurable arrays where various patterns are generated with common excitation magnitudes.

Many strategies have been proposed to cope with these non convex optimization problems. Notably, the flexibility of global optimization strategies has been extensively exploited [1]–[4] with the known drawback of the computational cost and without any guarantee regarding the optimality of the solution. Methods based on projection techniques have also been successfully applied to synthesize shaped beams [5], [6] and reconfigurable arrays with phase-only control [7], [8]. Finally, an original approach, that exploits by clever means the separate synthesis of pencil and shaped beam patterns, has been recently proposed in [9] to synthesize phase-only reconfigurable linear arrays.

With the recent advances in convex optimization, the SemiDefinite Relaxation (SDR) technique has lately shown a great significance and relevance on many applications in signal processing and communications as reviewed in [10]. The concept of SDR allows to relax the constraints of the original problem in order to formulate it as a convex and therefore easier to solve optimization problem. The SDR is thus an approximation technique for difficult optimization problems. This technique has been very recently successfully used to synthesize phased arrays with notches in the beam pattern [11], [12].

In this paper, a general procedure based on the SDR technique is developed and described to efficiently solve approximately various non convex array synthesis problems. The way to apply this powerful and computationally efficient approximation technique to the synthesis of shaped beams, arrays with phase-only control and reconfigurable arrays is detailed. The conditions under which the relaxed problems provide the optimal solution to the original non convex synthesis problems are specified. As shown in the numerical examples, it turns out that the proposed approach retrieves known optimal solutions in cases where these are known.

This paper is organized as follows. In Section II, the antenna array notations are introduced and the SDR technique is explained in the array synthesis context. The way to formulate the synthesis of shaped beams, arrays with phase-only control and reconfigurable arrays as convex optimization problems is detailed in Section III. A set of representative numerical examples are reported in Section IV to both validate and illustrate the proposed procedure. Conclusions are drawn in Section V.

II. Problem Formulation and Resolution

A. Antenna Array

Let us consider an array composed of N elements placed at locations $\vec{r}_n$ with $n = 1, \cdots, N$. For the sake of clarity, the problem is described for a one-dimensional pattern synthesis. The synthesis is performed over the polar angle θ in a fixed azimuthal plane $\varphi = \varphi_0$ that is omitted in the notations. The extension to a two-dimensional (2-D) pattern synthesis, i.e. a synthesis over both angular directions θ and φ, is straightforward. Each element n radiates a pattern $g_n(\theta)$ in the direction θ. The far field $f(\theta)$ radiated by the array is then:

$$f(\theta) = \boldsymbol{a}(\theta)^H \boldsymbol{w}, \qquad (1)$$
$$\text{with } \boldsymbol{a}(\theta) = \left[g_1(\theta)e^{j\frac{2\pi}{\lambda}\vec{r}_1.\hat{r}(\theta)} \cdots g_N(\theta)e^{j\frac{2\pi}{\lambda}\vec{r}_N.\hat{r}(\theta)}\right]^H$$

where $\boldsymbol{w}$ is the complex (magnitude and phase) excitation vector, $\hat{r}(\theta)$ is the unit vector in the direction θ and $.^H$ denotes

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the Hermitian transposition.
Let us introduce the notation $f_i = f(\theta_i)$ and $\boldsymbol{a}_i = \boldsymbol{a}(\theta_i)$. The real value version of (1) in the direction θ_i is then:

$$[\mathbb{R}(f_i)\ \ \mathbb{I}(f_i)]^T = \boldsymbol{A}_i\,\boldsymbol{x}, \qquad \text{with } \boldsymbol{A}_i = \begin{bmatrix} \mathbb{R}(\boldsymbol{a}_i{}^T) & -\mathbb{I}(\boldsymbol{a}_i{}^T) \\ \mathbb{I}(\boldsymbol{a}_i{}^T) & \mathbb{R}(\boldsymbol{a}_i{}^T) \end{bmatrix} \text{ and } \boldsymbol{x} = \begin{bmatrix} \mathbb{R}(\boldsymbol{w}) \\ \mathbb{I}(\boldsymbol{w}) \end{bmatrix} \tag{2}$$

where $\boldsymbol{A}_i \in \mathbb{R}^{2\times 2N}$, $\boldsymbol{x} \in \mathbb{R}^{2N\times 1}$, $.^T$ is the transpose operator, $\mathbb{R}$ and $\mathbb{I}$ stands for the real and imaginary parts respectively. The power radiated by the array is then:

$$\left|f_i\right|^2 = \boldsymbol{x}^T\boldsymbol{Q}_i\boldsymbol{x}, \text{ with } \boldsymbol{Q}_i = \boldsymbol{A}_i^T\boldsymbol{A}_i. \tag{3}$$

Note that in the formulation (3), the power radiated by arbitrary arrays, i.e. arrays of any given geometry and composed of elements with any known radiation patterns, can be considered.
Let us recall that for any real symmetric matrix $\boldsymbol{C}$ and any real vector $\boldsymbol{x}$:

$$\boldsymbol{x}^T\boldsymbol{C}\boldsymbol{x} = \mathrm{Tr}(\boldsymbol{x}^T\boldsymbol{C}\boldsymbol{x}) = \mathrm{Tr}(\boldsymbol{C}\boldsymbol{x}\boldsymbol{x}^T) \tag{4}$$

where $\mathrm{Tr}(\boldsymbol{A})$ is the trace (sum of the diagonal coefficients) of the matrix $\boldsymbol{A}$. The power (3) radiated by the array becomes:

$$\left|f_i\right|^2 = \mathrm{Tr}(\boldsymbol{Q}_i\boldsymbol{X}), \text{ with } \boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^T \in \mathbb{R}^{2N\times 2N}. \tag{5}$$

At this step, it is important to observe that $\boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^T$ is equivalent to $\boldsymbol{X}$ being a symmetric positive semidefinite matrix (denoted $\boldsymbol{X} \succeq 0$) of rank one ($\mathrm{rank}(\boldsymbol{X}) = 1$).

B. The concept of SemiDefinite Relaxation

Let us consider a typical array synthesis problem in order to explain the SDR technique. Many synthesis problems amount to look for the array excitations $\boldsymbol{x}$ such that the power radiated by the array is constrained or equivalently $|f_i|^2$ belongs to a set $\mathcal{C}_i$ for the directions $i = 1, ..., I$. With (3), such problem can be formulated as follows:

$$\text{find } \boldsymbol{x} \text{ such that } \boldsymbol{x}^T\boldsymbol{Q}_i\boldsymbol{x} \in \mathcal{C}_i, \text{ for } i = 1, ..., I. \tag{6}$$

Using (4) and (5), the problem (6) is equivalent to:

$$\text{find } \boldsymbol{X} \text{ such that } \left\{ \begin{array}{l} \mathrm{Tr}(\boldsymbol{Q}_i\boldsymbol{X}) \in \mathcal{C}_i, \text{ for } i = 1, ..., I \\ \boldsymbol{X} \succeq 0 \\ \mathrm{rank}(\boldsymbol{X}) = 1 \end{array} \right. . \tag{7}$$

The problem (7) is not convex because of the rank constraint. By dropping this constraint, we obtain the following relaxation of (7):

$$\text{find } \boldsymbol{X} \text{ such that } \left\{ \begin{array}{l} \mathrm{Tr}(\boldsymbol{Q}_i\boldsymbol{X}) \in \mathcal{C}_i, \text{ for } i = 1, ..., I \\ \boldsymbol{X} \succeq 0 \end{array} \right. \tag{8}$$

that is called SemiDefinite Relaxation (SDR) since it is an instance of semidefinite programming. The convex formulation (8) is convenient because it can be solved optimally by readily available software such as CVX [13].
Of course, there is a price to pay in turning the NP-hard problem (6) into the polynomial-time solvable problem (8). The main issue is indeed to transform the globally optimal solution $\boldsymbol{X}^*$ of the SDR (8) into a feasible point $\tilde{\boldsymbol{x}}$ of the original synthesis problem (6). If $\mathrm{rank}(\boldsymbol{X}^*) = 1$, then $\boldsymbol{X}^* = \boldsymbol{x}^*\boldsymbol{x}^{*T}$ and $\boldsymbol{x}^*$ is not only a feasible point but also the optimal solution of (6). However, standard interior point methods solving semidefinite programs do not necessarily return a low-rank solution and, in general, the solution $\boldsymbol{X}^*$ of (8) is such as $\mathrm{rank}(\boldsymbol{X}^*) > 1$.
To encourage low-rank solutions, several techniques reviewed in [14] have been proposed. A well known convex heuristic is to minimize the trace of $\boldsymbol{X}$ which amounts to minimize the sum of the eigenvalues of $\boldsymbol{X}$ and therefore its rank. Specifically, the reweighted minimization algorithm, detailed in [14], can be used. At each step k, the following convex optimization problem is solved:

$$\begin{array}{ll} \displaystyle\min_{\boldsymbol{X}^k} & \mathrm{Tr}\left((\boldsymbol{X}^{k-1} + \delta\mathbf{I})^{-1}\boldsymbol{X}^k\right) \\ \text{subject to} & \left\{ \begin{array}{l} \mathrm{Tr}(\boldsymbol{Q}_i\boldsymbol{X}^k) \in \mathcal{C}_i, \text{ for } i = 1, ..., I \\ \boldsymbol{X}^k \succeq 0 \end{array} \right. \end{array} \tag{9}$$

where δ can be seen as a small regularization constant, $\mathbf{I}$ is the identity matrix and $\boldsymbol{X}^0 = \mathbf{I}$. Nevertheless, the procedure (9) does not ensure the obtention of a rank one solution $\boldsymbol{X}^*$ of (8).
When $\mathrm{rank}(\boldsymbol{X}^*) > 1$, one intuitive way to extract a vector $\tilde{\boldsymbol{x}}$ that is feasible for (6) is to apply a rank one approximation $\boldsymbol{X}_1^*$ of $\boldsymbol{X}^*$. The best one, in the least two norm sense, can be obtained via a eigenvalue decomposition [15] as follows:

$$\boldsymbol{X}_1^* = \sigma_1\boldsymbol{u}_1\boldsymbol{u}_1{}^T, \tag{10}$$

where σ_1 is the largest eigenvalue of $\boldsymbol{X}^*$ and $\boldsymbol{u}_1$ is the corresponding eigen vector. The vector $\tilde{\boldsymbol{x}} = \sqrt{\sigma_1}u_1$ is then a potential solution of (6) provided that it is a feasible solution.
Finally, it is important to point out that even though the extracted solution $\tilde{\boldsymbol{x}}$ is feasible for (6), there is no guarantee that it is an optimal solution. For otherwise, it would mean that we have solved a NP-hard problem in a polynomial time.

III. Application of SemiDefinite Relaxation to Array Synthesis Problems

The SDR technique described in Section II is here applied to three kinds of non convex array synthesis problems.

A. Shaped Beam Synthesis

The synthesis of shaped beams generally requires to find the array excitations to generate a power pattern complying to a given mask. For this mask feasibility problem, the power radiated by the array is typically:

- upper and lower bounded by $u(\theta)$ and $l(\theta)$ respectively over an angular region SB (Shaped Beam),
- upper bounded by an envelope $\rho(\theta)$ over the region SL (SideLobe).

These constraints can be formulated:

$$\left\{ \begin{array}{rl} l(\theta) \le \left|f(\theta)\right|^2 \le u(\theta), & \text{for } \theta \in \text{SB} \\ \left|f(\theta)\right|^2 \le \rho(\theta), & \text{for } \theta \in \text{SL} \end{array} \right. \tag{11}$$

which yields respectively after discretization:

$$\left\{ \begin{array}{rl} l_m \le \left|f_m\right|^2 \le u_m, & \text{for } m = 1..., M \\ \left|f_q\right|^2 \le \rho_q, & \text{for } q = 1..., Q \end{array} \right. \tag{12}$$

where $f_i = f(\theta_i)$.
Using (3), the shaped beam synthesis problem becomes:

$$\text{find } \boldsymbol{x} \text{ such that } \begin{cases} \boldsymbol{x}^T\boldsymbol{Q}_m\boldsymbol{x} \geq l_m, & \text{for } m = 1..., M \\ \boldsymbol{x}^T\boldsymbol{Q}_m\boldsymbol{x} \leq u_m, & \text{for } m = 1..., M \\ \boldsymbol{x}^T\boldsymbol{Q}_q\boldsymbol{x} \leq \rho_q, & \text{for } q = 1..., Q \end{cases}. \tag{13}$$

According to (8), the SDR of (13) is:

$$\text{find } \boldsymbol{X} \text{ such that} \tag{14}$$
$$\boldsymbol{X} \in \mathcal{S} : \begin{cases} \mathrm{Tr}(\boldsymbol{Q}_m\boldsymbol{X}) \geq l_m, & \text{for } m = 1..., M \\ \mathrm{Tr}(\boldsymbol{Q}_m\boldsymbol{X}) \leq u_m, & \text{for } m = 1..., M \\ \mathrm{Tr}(\boldsymbol{Q}_q\boldsymbol{X}) \leq \rho_q, & \text{for } q = 1..., Q \end{cases}$$
$$\text{with } \boldsymbol{X} \succeq 0.$$

The set of constraints $\mathcal{S}$ defines the shaped beam problem.
As detailed at the end of Section II-B, the procedure (9) is used to find a rank one matrix $\boldsymbol{X}$ that satifies (14). If $\mathrm{rank}\,\boldsymbol{X} > 1$, the approximation (10) is used to find an excitation vector $\tilde{\boldsymbol{x}}$ that is a feasible solution of the original shaped beam synthesis problem (13).

B. Phase-Only Synthesis

For the synthesis of arrays with phase-only control, the excitation magnitudes are fixed and known ($|w_n|^2 = \alpha_n,$ for $n = 1, ..., N$) while the phases are left free. Let us remind that the excitation vector $\boldsymbol{x}$ is:

$$\boldsymbol{x} = \left[\mathbb{R}(w_1)\cdots\mathbb{R}(w_N)\ \ \mathbb{I}(w_1)\cdots\mathbb{I}(w_N)\right]^T \in \mathbb{R}^{2N}. \tag{15}$$

The excitation magnitudes can be expressed as follows:

$$\alpha_n = |w_n|^2 = \boldsymbol{x}^T\boldsymbol{Q}_n\boldsymbol{x}, \quad n = 1, ..., N. \tag{16}$$

where $\boldsymbol{Q}_n$ are N diagonal matrices of dimension $2N \times 2N$:

$$\boldsymbol{Q}_n(i,i) = \begin{cases} 1 & \text{if } i = n \\ 1 & \text{if } i = n + N \\ 0 & \text{elsewhere} \end{cases}. \tag{17}$$

Using (4), the excitations magnitudes (16) can be written:

$$|w_n|^2 = \mathrm{Tr}(\boldsymbol{Q}_n\boldsymbol{X}) \text{ where } \boldsymbol{X} \succeq 0 \text{ and } \mathrm{rank}(\boldsymbol{X}) = 1. \tag{18}$$

The synthesis of an array with phase-only control that radiates a pattern defined by a set of constraints $\mathcal{C}$ is then:

$$\text{find } \boldsymbol{X} \text{ such that } \begin{cases} \mathrm{Tr}(\boldsymbol{Q}_n\boldsymbol{X}) = \alpha_n, & n = 1, ..., N \\ \boldsymbol{X} \in \mathcal{C} \\ \boldsymbol{X} \succeq 0 \text{ and } \mathrm{rank}(\boldsymbol{X}) = 1 \end{cases} \tag{19}$$

where the excitation magnitude α_n are set as desired. The set of constraints $\mathcal{C}$ is defined by:

- $\mathcal{S}$ in (14) to generate a shaped beam,
- $\mathcal{F}$ in (22) of Appendix I to generate a focused beam,
- $\mathcal{D}$ in (24) of Appendix II to generate a difference pattern.

To solve (19), the SDR technique is applied i.e. the rank constraint is dropped. The procedure detailed at the end of Section II-B is then used to find the excitation vector $\boldsymbol{x}$ solution of the phase-only synthesis problem.

C. Synthesis of Reconfigurable Array by Phase-Only Control

A single array can be used to radiate more than one pattern. This array is reconfigurable by phase-only control when the switch between patterns is carried out by modifying only the excitation phases. A simple and efficient procedure to synthesize reconfigurable arrays by phase-only control, i.e. to determine simultaneously both the common excitation amplitudes and the various phases, is proposed.
For the sake of simplicity, let us detail the procedure for a reconfigurability between two patterns. The extension to more than two patterns is straightforward. Each pattern j is defined by a set of constraints $\mathcal{C}_j$. The synthesis problem amounts to look for the excitation vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ of same magnitude that generate $\mathcal{C}_1$ and $\mathcal{C}_2$. Using (16) and (17), the equality of the excitation magnitudes is enforced by setting ${\boldsymbol{x}_1}^T\boldsymbol{Q}_n\boldsymbol{x}_1 = {\boldsymbol{x}_2}^T\boldsymbol{Q}_n\boldsymbol{x}_2$ or equivalently:

$$\boldsymbol{x}^T\boldsymbol{Q}_{n_1}\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{Q}_{n_2}\boldsymbol{x} \quad \text{with } \boldsymbol{x}^T = [{\boldsymbol{x}_1}^T{\boldsymbol{x}_2}^T] \in \mathbb{R}^{1\times 4N}$$

where $\boldsymbol{Q}_{n_1}$ and $\boldsymbol{Q}_{n_2}$ are $2N$ diagonal matrices of dimension $4N \times 4N$:

$$\boldsymbol{Q}_{n_1}(i,i) = \begin{cases} 1 \text{ if } i = n \\ 1 \text{ if } i = n + N \\ 0 \text{ elsewhere} \end{cases}$$
$$\text{and } \boldsymbol{Q}_{n_2}(i,i) = \begin{cases} 1 \text{ if } i = n + 2N \\ 1 \text{ if } i = n + 3N \\ 0 \text{ elsewhere} \end{cases}.$$

These constraints can be easily modified in case not all but only a few excitation magnitudes are common between the two patterns.
The synthesis of an array that is reconfigurable between $\mathcal{C}_1$ and $\mathcal{C}_2$ by phase-only control is formulated as follows:

$$\text{find } \boldsymbol{X} \text{ such that } \begin{cases} \mathrm{Tr}(\boldsymbol{Q}_{n_1}\boldsymbol{X}) = \mathrm{Tr}(\boldsymbol{Q}_{n_2}\boldsymbol{X}) \\ \boldsymbol{X} \in \mathcal{C}_1 \\ \boldsymbol{X} \in \mathcal{C}_2 \\ \boldsymbol{X} \succeq 0 \text{ and } \mathrm{rank}(\boldsymbol{X}) = 1 \end{cases}. \tag{20}$$

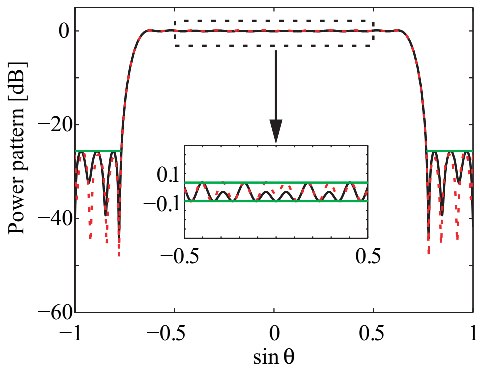
The set $\mathcal{C}_i$ is either equal to $\mathcal{S}$, $\mathcal{F}$ or $\mathcal{D}$ to generate a shaped beam, a focused beam or a difference pattern respectively.
The SDR technique and matrix decomposition described at the end of Section II-B are then applied to retrieve the excitation vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

IV. Numerical Results

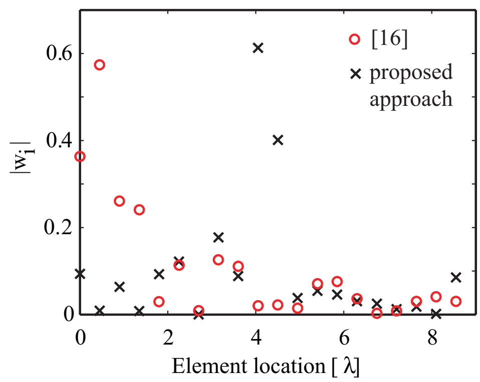
Various examples of shaped beam synthesis and synthesis of reconfigurable array with phase-only control are presented to both validate and illustrate the potentialities of the approach presented in Sections II and III.

A. Shaped Beam Synthesis

1) Sectoral Pattern Synthesis: The synthesis of a shaped beam with a linear array composed of 20 isotropic elements that are uniformly spaced ($0.45\,\lambda$) is addressed. The goal is to achieve a sectoral power pattern with a ripple of $\pm 0.1\,$dB in the shaped beam region (SB) defined by $|\theta| \leq 40\,^\circ$ and to minimize the sidelobe level for angles such that $|\theta| \geq 50\,^\circ$.
This specific problem has an optimal solution obtained via



(a)



(b)

Fig. 1. Sectoral shaped beam synthesis of a 20 element linear array. (a) Far field pattern obtained by the proposed approach (solid line) and by the reference optimal method [16] in dashed line. (b) Array element excitations obtained by [16] and the proposed approach.

spectral factorization in [16] that is used as a reference to assess the proposed method. The synthesized far field patterns and the corresponding excitations are given in Fig. 1(a) and (b) respectively. The radiation performances obtained by the proposed approach are very close to the optimal ones (same sidelobe level and similar shaped beam ripple) even if the excitations are quite different.

2) Cosecant Pattern Synthesis: The synthesis of a cosecant beam with a linear array composed of 30 isotropic elements that are half wavelength spaced is considered. An heuristic procedure (tabu search algorithm) has been used to find the array excitations in [1]. The proposed approach also manages to find a solution that satisfies this stringent far field template. The far field patterns and array element excitations are plotted in Fig. 2.

B. Synthesis of Reconfigurable Array by Phase-only Control

1) Focused Beam - Shaped Beam Synthesis: Let us consider an example of reconfigurable array synthesis that is presented in [9]. The goal is to determine the complex excitations of a linear equispaced array composed of 20 half wavelength spaced isotropic elements. By only changing the phases of the excitations, the pattern radiated by the array must switch from a focused to a shaped sectoral beam and vice versa.
The constraints on the power patterns (see the blue dashed lines plotted in Fig. 3(a)) are the following:
- for the focused beam, a sidelobe level below -27.45 dB for $|\sin\theta| \geq 0.15$,
- for the sectoral beam, a sidelobe level below -25.5 dB for $|\sin\theta| \geq 0.35$ and a shaped beam ripple of ± 0.43 dB over $|\sin\theta| \leq 0.2$.

The synthesized far field patterns and array element excitations are plotted in Fig. 3. The proposed approach allows to determine at once (in less than 30 s on a standard laptop) both the common excitation magnitudes and different phases to generate both the focused and shaped beam patterns.

2) Shaped Beam - Shaped Beam Synthesis: Let us consider an array composed of thirty half wavelength spaced isotropic elements. The goal is to find the two sets of excitations of common magnitudes, such that the array can switch between a sectoral and a cosecant beam by only changing the phases. In each case, the ripple of the shaped beam is of ± 0.5 dB and the sidelobes are below -15 dB.
The synthesis results of the reconfigurable array by phase-only control are plotted in Fig. 4. The proposed approach allows to determine at once both the common excitation magnitudes and different phases to generate a sectoral and a cosecant beam.

3) Focused Beam (Sum) - Difference Pattern Synthesis: The synthesis of a reconfigurable linear array composed of ten half wavelength spaced isotropic elements is addressed. The requirements provided in the first numerical example of [18] are followed. Six excitation amplitudes are shared to switch from a focused beam (also known as sum pattern) to a difference pattern. The radiating constraints are the following:
- for the sum pattern, a beamwidth (null to null) of 30.4 ° and sidelobes below -24 dB,
- for the difference pattern, a beamwidth of 52 ° and sidelobes below -18.8 dB.

The proposed approach recovers the optimal results given in [18]. The radiation patterns and excitations are shown in Fig. 5 and Table I respectively.

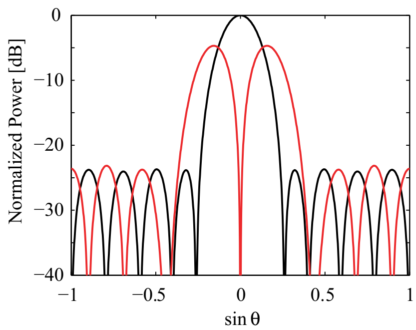


Fig. 5. Synthesis of reconfigurable array with common excitation magnitudes: sum and difference patterns.

V. Conclusion

A general procedure has been developed and described to approximately solve a wide range of difficult, because not

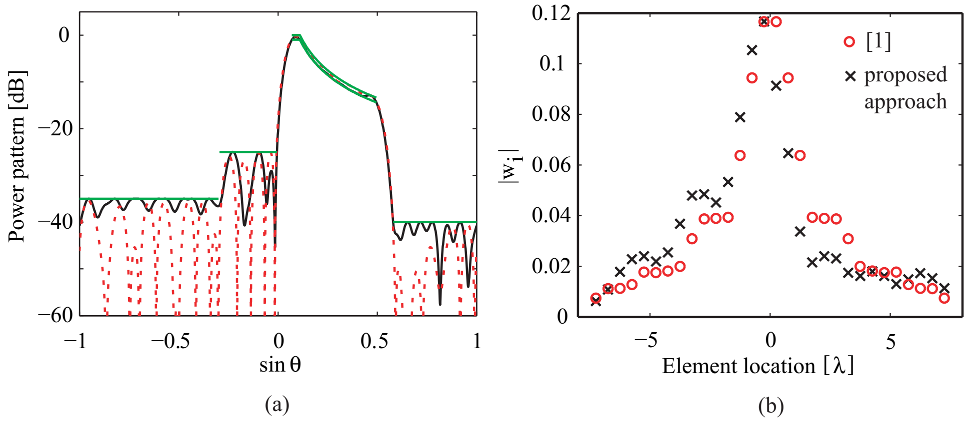


Fig. 2. Cosecant beam synthesis of a 30 element linear array. (a) Far field pattern obtained by the proposed approach (solid line) and the global optimization method in [1] (dashed line) with (b) the corresponding array element excitation magnitudes.

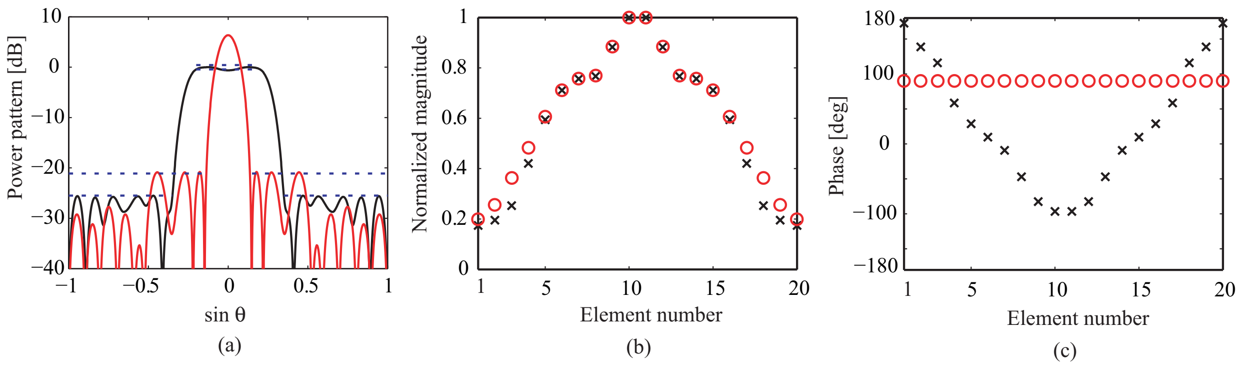


Fig. 3. Synthesis results of reconfigurable array by phase only control: (a) focused beam and sectoral far field radiation pattern with the corresponding element excitation (b) magnitudes and (c) phases.

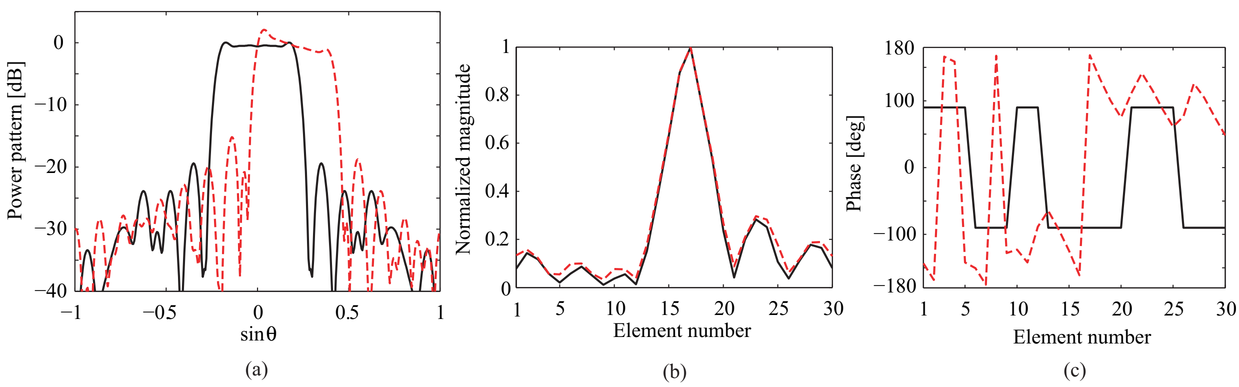


Fig. 4. Synthesis results of reconfigurable array by phase only control: (a) sectoral and cosecant far field radiation pattern with the corresponding element excitation (b) magnitudes and (c) phases.

convex, array synthesis problems. The constraints of these problems are relaxed in order to transform the array synthesis into a convex optimization problem. The way to apply the convex relaxation to the synthesis of shaped beams, arrays with phase-only control and reconfigurable array with common excitation magnitudes is detailed. This powerful approximation technique, known as semidefinite relaxation technique, provides, under certain conditions given in the paper, the optimal solution of the original non convex synthesis problem.

The advantages of the proposed procedure over competing methods are manifold. First, it is versatile and can deal with a wide range of synthesis problems with only a small modification of the constraints as shown in the paper. Second, the method is easy to implement and there is no parameter to be tuned. Moreover, it is computationally effective and only calls for freely available routines. Finally, there is no restriction regarding the type of array and pattern to be synthesized. Indeed, arbitrary arrays and any beam patterns (focused or

TABLE I
NORMALIZED EXCITATIONS FOR THE FOCUSED BEAM (SUM) - DIFFERENCE PATTERN SYNTHESIS

n	w_{FB}	w_{DP}
1	0.4388	-0.4388
2	0.5252	-0.5252
3	0.7338	-0.7338
4	0.8993	-0.6906
5	1.0000	-0.2806
6	1.0000	0.2806
7	0.8993	0.6906
8	0.7338	0.7338
9	0.5252	0.5252
10	0.4388	0.4388

shaped beam pattern with arbitrary sidelobe envelope, difference patterns, etc...) can be handled.

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APPENDIX I
SYNTHESIS OF FOCUSED BEAM PATTERN

The synthesis of a focused beam pattern can be formulated in a convex way as shown in [19], [20]. This problem amounts to find the array excitations such that the constraints $\mathcal{F}$ are satisfied:

$$\mathcal{F}: \left\{ \begin{array}{l} f(\theta_0)^2 \geq 1 \\ f(\theta)^2 \leq \rho(\theta), \text{ for } \theta \in SL \end{array} \right. \tag{21}$$

where θ_0 is the main beam direction. The synthesis of a focused beam pattern amounts to find $\boldsymbol{X}$ such that $\boldsymbol{X} \in \mathcal{F}$ with:

$$\mathcal{F}: \left\{ \begin{array}{l} \mathrm{Tr}(\boldsymbol{Q}_0\boldsymbol{X}) \geq 1, \\ \mathrm{Tr}(\boldsymbol{Q}_q\boldsymbol{X}) \leq \rho_q, \text{ for } q = 1..., Q \end{array} \right. \tag{22}$$

with $\boldsymbol{X} \succeq 0$ and $\mathrm{rank}(\boldsymbol{X}) = 1$.

APPENDIX II
SYNTHESIS OF DIFFERENCE PATTERN

The synthesis of a difference pattern can be formulated in a convex way as shown in [21]. This problem amounts to find the array excitations such that the following constraints are satisfied:

$$\mathcal{D}: \left\{ \begin{array}{l} \left.\dfrac{\partial f(\theta)^2}{\partial \theta}\right|_{\theta=\theta_0} \geq \gamma \\ f(\theta_0)^2 = 0 \\ f(\theta)^2 \leq \rho(\theta), \text{ for } \theta \in SL \end{array} \right. \tag{23}$$

where γ is a constant to ensure an important slope in the direction θ_0 where the target is (see [21] for more details). The synthesis of a difference pattern amounts to find $\boldsymbol{X}$ such that $\boldsymbol{X} \in \mathcal{D}$ with:

$$\mathcal{D}: \left\{ \begin{array}{l} \mathrm{Tr}(\boldsymbol{Q}_{d_0}\boldsymbol{X}) \geq \gamma' \\ \mathrm{Tr}(\boldsymbol{Q}_0\boldsymbol{X}) = 0, \\ \mathrm{Tr}(\boldsymbol{Q}_q\boldsymbol{X}) \leq \rho_q, \text{ for } q = 1..., Q \end{array} \right. \tag{24}$$

with $\boldsymbol{X} \succeq 0$ and $\mathrm{rank}(\boldsymbol{X}) = 1$.

where the diagonal matrix $\boldsymbol{Q}_{d_0}$ of dimension $2N \times 2N$ is such that:

$$\boldsymbol{Q}_{d_0}(i,i) = \left\{ \begin{array}{ll} |\vec{r}_i|, & \text{if } i = 1, ..., N \\ |\vec{r}_{i-N}|, & \text{if } i = N+1, ..., 2N \end{array} \right. .$$

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